Combinatorics and Representation Theory

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$$3! = 1^2 + 2^2 + 1^2$$
$$4! = 1^2 + 3^2 + 2^2 + 3^2 + 1^2$$
$$5! = 1^2 + 4^2 + 5^2 + 6^2 + 5^2 + 4^2 + 1^2$$

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Groups

A group G has a set S of elements and an operation $\times$, and respects these axioms:

- Closure: if a and b are in S, then $a \times b$ is also in S.
- Associativity: for any a, b, and c in S, $(ab)c = a(bc)$.
- Identity: there exists an identity element e in S such that for all a in G, $ae = ea = a$.
- Inverse: for all a in G, there exists an inverse b such that $ab = ba = e$.

Some examples of groups are

- $\{\mathbb{Z}, +\}$
- $\{\mathbb{R}, \cdot\}$

Symmetric groups

A symmetric group S_n is a group with bijective functions $f : \{1, 2, ..., n\} \to \{1, 2, ..., n\}$ as the elements and function composition as its operation.

For example, the elements of S_3 are

(i) $1 \mapsto 1$, $2 \mapsto 2$, $3 \mapsto 3$ (the identity map)

(ii) $1 \mapsto 1$, $2 \mapsto 3$, $3 \mapsto 2$,

(iii) $1 \mapsto 2$, $2 \mapsto 1$, $3 \mapsto 3$,

(iv) $1 \mapsto 2$, $2 \mapsto 3$, $3 \mapsto 1$,

(v) $1 \mapsto 3$, $2 \mapsto 1$, $3 \mapsto 2$,

(vi) $1 \mapsto 3$, $2 \mapsto 2$, $3 \mapsto 1$.

The elements of S_n are called permutations.

Line notation (two and one)

In two-line notation...

- The first line is the elements of the domain in numerical order.
- The second line is the elements of the range that correspond with the elements of the domain.

In one-line notation, the first line is deleted and only the second line is written.

For example, here is some two-line notation and the corresponding one-line notation.

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 3 & 5 & 4 & 1 & 2 & 6 \end{pmatrix}$$

$$354126$$

Cycle notation

In cycle notation, the first cycle starts at 1 : the 1 is written down, then the number that 1 maps to, then the number that that number maps to is written down, and so on until the 1 is mapped back to.

That forms a cycle that is placed in parentheses. The second cycle starts at the next smallest and available number, and so on until every element of the set is in a cycle.

For example, here is the previous two-line notation and the corresponding cycle notation.

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 3 & 5 & 4 & 1 & 2 & 6 \end{pmatrix}$$

$$(134)(25)(6).$$

Sign of a permutation

Two ways to find the sign of a permutation are

- When written in one-line notation, it is the parity of the number of pairs of values that are out of order.
- When written in cycle notation as a product of 2-cycles, it is the parity of the number of cycles.

Multiplication (AKA Composition) of Permutations

The first permutation is written, then for each number, the number that the first permutation maps to in the second permutation is written. This can be done multiple times, and the last line is the final product.

For example, the product of 2431 and 2314 is

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 3 & 1 \\ 3 & 4 & 1 & 2 \end{pmatrix}$$

which simplifies to

$$3412.$$

We can also multiply these permutations in cycle notation. 2431 is $(1\ 2\ 4)(3)$ in cycle notation, and 2314 is $(1\ 2\ 3)(4)$ in cycle notation. Their product is $(1\ 2\ 3)(4)(1\ 2\ 4)(3) = (1\ 3)(2\ 4)$.

Partitions

A partition $(\lambda_1, \lambda_2, \ldots, \lambda_k)$ of n is a k tuple that sums to n.

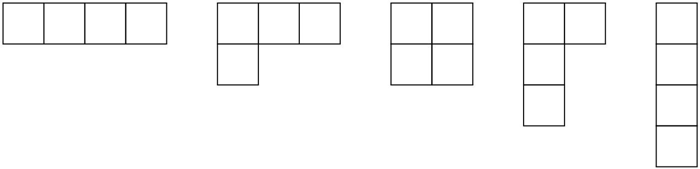
For example, the partitions of 4 are:

- (4)
- (3, 1)
- (2, 2)
- (2, 1, 1)
- (1, 1, 1, 1)

Young diagrams

Given a partition $(\lambda_1, \lambda_2, \ldots, \lambda_k)$, a Young diagram is a collection of $\sum_i \lambda_i$ boxes so that row i has λ_i boxes.

For example, the same partitions of 4 illustrated as Young diagrams are:



Note that by convention, the boxes are left-justified.

Standard Young tableaux

Given a partition $(\lambda_1, \lambda_2, \ldots, \lambda_k)$, a standard Young tableaux (SYT) is a diagram with k rows with $\lambda_1, \lambda_2, \ldots, \lambda_k$ boxes, respectively. Each box is assigned with exactly one integer from 1 to n without repeats such that the values increase top to bottom in columns and left to right across rows.

For example, the partition $(2, 2)$ of 4 has 2 standard Young tableaux:

1	3
2	4

1	2
3	4

Reflections of standard Young tableaux

And the partition $(3, 1)$ of 4 has 3 standard Young tableaux:

$$\begin{array}{|c|c|c|}\hline 1 & 2 & 3 \\ \hline 4 \\ \cline{1-1} \end{array} \qquad \begin{array}{|c|c|c|}\hline 1 & 2 & 4 \\ \hline 3 \\ \cline{1-1} \end{array} \qquad \begin{array}{|c|c|c|}\hline 1 & 3 & 4 \\ \hline 2 \\ \cline{1-1} \end{array}$$

The partition $(2, 1, 1)$ of 4 also has 3 standard Young tableaux:

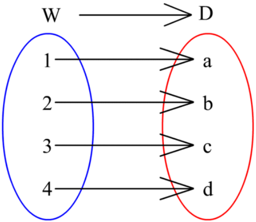
$$\begin{array}{|c|c|}\hline 1 & 4 \\ \hline 2 \\ \cline{1-1} 3 \\ \cline{1-1} \end{array} \qquad \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 \\ \cline{1-1} 4 \\ \cline{1-1} \end{array} \qquad \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 \\ \cline{1-1} 4 \\ \cline{1-1} \end{array}$$

In general, if we reflect any standard Young tableaux over the top-left to bottom-right diagonal, then we get another standard Young tableaux.

Robinson-Schensted Correspondence

Definition

The Robinson-Schensted Correspondence is a bijection between permutations and pairs of Standard Young Tableaux of the same size. The two Standard Young Tableaux are called the Insertion Tableaux, P, and the Recording Tableaux, Q.



1	3	4
2	5	

and

1	2	4
3	5	

$\leftrightarrow$ (123)(45)

So How Does It Work?

Suppose we want to create the two Standard Young Tableaux for a permutation with one-line notation

$$21453$$

We begin by finding the Insertion Tableaux using the Schensted Algorithm. We read the permutation left to right, inserting number using special rules so that the tableaux increases as it goes down and increases as it goes right. First, we write the first number, which is 2.

$$\boxed{2}$$

Rule i: A New Minimum

For our permutation 21453, our next number is 1.

$$\begin{array}{|c|}\hline 2 \\ \hline \end{array} \qquad \begin{array}{|c|}\hline 1 \\ \hline 2 \\ \hline \end{array}$$

Rule ii: Larger Than Every Number In The Top Row

For our permutation 21453, our next number is 4. Then, we add a 5 by the same process.

2

1
2

1	4
2	

1	4	5
2		

Rule iii: Neither Smallest Nor Largest Top Row Value

For our permutation 21453, our next number is 3.

$$
\begin{array}{|c|}\hline 2 \\ \hline \end{array}
\qquad
\begin{array}{|c|}\hline 1 \\ \hline 2 \\ \hline \end{array}
\qquad
\begin{array}{|c|c|}\hline 1 & 4 \\ \hline 2 & \multicolumn{1}{c}{} \\ \cline{1-1} \end{array}
\qquad
\begin{array}{|c|c|c|}\hline 1 & 4 & 5 \\ \hline 2 & \multicolumn{2}{c}{} \\ \cline{1-1} \end{array}
\qquad
\begin{array}{|c|c|c|}\hline 1 & 3 & 5 \\ \hline 2 & 4 & \multicolumn{1}{c}{} \\ \cline{1-2} \end{array}
$$

Summary (So Far) Of The Schensted Algorithm

- **A New Minimum:** Put the new smallest number in the top left corner, and shift the leftmost column down.
- **Larger Than Every Number In The Top Row:** Put the new number on the rightmost end of the top row.
- **Everything Else:** Put the number into the top row so that the row is still increasing, and then shift the new number's column down. Bump if necessary.

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This algorithm is well-defined and each rule maintains the "standardness" - rows are increasing left to right, and columns are increasing top to bottom.

But the Schensted Algorithm produces **a pair** of Standard Young Tableaux! How do you get the second Standard Young Tableaux?

Recording Tableaux

To get the Recording Tableaux, we can invert the permutation and use the same process. The inverse of 21453 is 21534.

$$\begin{array}{|c|}\hline 2 \\ \hline\end{array} \qquad \begin{array}{|c|}\hline 1 \\ \hline 2 \\ \hline\end{array} \qquad \begin{array}{|c|c|}\hline 1 & 5 \\ \hline 2 \\ \cline{1-1}\end{array} \qquad \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 & 5 \\ \hline\end{array} \qquad \begin{array}{|c|c|c|}\hline 1 & 3 & 4 \\ \hline 2 & 5 \\ \cline{1-2}\end{array}$$

Another Way To Get Recording Tableaux

If you construct the Insertion Tableaux, and then separately write down the order that each cell is filled in, you will also get the Recording Tableaux.

$$\begin{array}{|c|}\hline 2 \\ \hline \end{array} \qquad \begin{array}{|c|}\hline 1 \\ \hline 2 \\ \hline \end{array} \qquad \begin{array}{|c|c|}\hline 1 & 4 \\ \hline 2 \\ \cline{1-1} \end{array} \qquad \begin{array}{|c|c|c|}\hline 1 & 4 & 5 \\ \hline 2 \\ \cline{1-1} \end{array} \qquad \begin{array}{|c|c|c|}\hline 1 & 3 & 5 \\ \hline 2 & 4 \\ \cline{1-2} \end{array}$$

Here is the order we filled the Insertion Tableaux cells in!

$$\begin{array}{|c|}\hline 1 \\ \hline \end{array} \qquad \begin{array}{|c|}\hline 1 \\ \hline 2 \\ \hline \end{array} \qquad \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 \\ \cline{1-1} \end{array} \qquad \begin{array}{|c|c|c|}\hline 1 & 3 & 4 \\ \hline 2 \\ \cline{1-1} \end{array} \qquad \begin{array}{|c|c|c|}\hline 1 & 3 & 4 \\ \hline 2 & 5 \\ \cline{1-2} \end{array}$$

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$$\begin{array}{|c|}\hline 2 \\ \hline\end{array} \qquad \begin{array}{|c|}\hline 1 \\ \hline 2 \\ \hline\end{array} \qquad \begin{array}{|c|c|}\hline 1 & 4 \\ \hline 2 & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array} \qquad \begin{array}{|c|c|c|}\hline 1 & 4 & 5 \\ \hline 2 & \multicolumn{2}{c}{} \\ \cline{1-1}\end{array} \qquad \begin{array}{|c|c|c|}\hline 1 & 3 & 5 \\ \hline 2 & 4 & \multicolumn{1}{c}{} \\ \cline{1-2}\end{array}$$

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$$\begin{array}{|c|}\hline 1 \\ \hline\end{array} \qquad \begin{array}{|c|}\hline 1 \\ \hline 2 \\ \hline\end{array} \qquad \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array} \qquad \begin{array}{|c|c|c|}\hline 1 & 3 & 4 \\ \hline 2 & \multicolumn{2}{c}{} \\ \cline{1-1}\end{array} \qquad \begin{array}{|c|c|c|}\hline 1 & 3 & 4 \\ \hline 2 & 5 & \multicolumn{1}{c}{} \\ \cline{1-2}\end{array}$$

Because of this, a permutation maps to two Standard Young Tableaux of the same shape.

General Linear Group

Definition

A general linear group is the set of all $n \times n$ matrices with a non-zero determinant under the group operation of matrix multiplication. The general linear group we will use is $GL(n, \mathbb{C})$, where the matrix entries are complex numbers.

$\left[\frac{1}{2} + i\frac{\sqrt{3}}{2}\right]$ would be an element of $GL(1, \mathbb{C})$.

- Very useful in Representation Theory - Cayley's Theorem allows us to represent every element of a finite group as a matrix

Homomorphisms

Definition

A homomorphism is a mapping $\phi : G \to H$ that preserves the group operation. Therefore, for all $g_1, g_2 \in G, \phi(g_1 g_2) = \phi(g_1)\phi(g_2)$.

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Example

Take group $GL(2, \mathbb{C})$.
Then, the map $\phi : G \to \mathbb{C}^*$ by $\phi(n) = \det(g)$ where $g \in G$ is defined as a homomorphism, since for all elements $g_1, g_2 \in G$,
$\phi(g_1 g_2) = \det(g_1 g_2) = \det(g_1)\det(g_2) = \phi(g_1)\phi(g_2)$.

Remark

By convention, "$\mathbb{C}^*$" is used to denote the nonzero complex numbers.

Complex Representations of a Finite Group

Definition

An example of a representation of a group G is a homomorphism $\rho : G \to GL(n, \mathbb{C})$ where GL is a general linear group with $n \times n$ matrices with non-zero determinant and complex entries.

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Definition

The dimension of representation ρ is n.

Representations of S_3

We will define our functions $\rho_1 : S_3 \to \mathrm{GL}(1, \mathbb{C})$, $\rho_2 : S_3 \to \mathrm{GL}(1, \mathbb{C})$, $\rho_3 : S_3 \to \mathrm{GL}(2, \mathbb{C})$

- $\rho_1(g) = 1$ (the trivial representation),
- $\rho_2(g) = sgn(g)$ (the sign representation),
- $\rho_3(g)$: $e \mapsto \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $(1\ 2) \mapsto \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $(1\ 3) \mapsto \begin{pmatrix} -1 & 0 \\ -1 & 1 \end{pmatrix}$, $(2\ 3) \mapsto \begin{pmatrix} 1 & -1 \\ 0 & -1 \end{pmatrix}$, $(1\ 2\ 3) \mapsto \begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix}$, $(1\ 3\ 2) \mapsto \begin{pmatrix} -1 & 1 \\ -1 & 0 \end{pmatrix}$.

Then,

- χ_1: $\chi_1(g) = 1$ for all $g \in S_3$.
- χ_2: $\chi_2(g) = 1$ if g is even and -1 if g is odd.
- χ_3: $\chi_3(e) = 2$, $\chi_3(g) = 0$ if g is a 2-cycle, $\chi_3(g) = -1$ if g is a 3-cycle.

Character Tables

To visualize these functions and the characters, we can create something called a *character table*.

Using our example from before,

	Conjugacy Classes		
Representations	e	(a b)	(a b c)
χ_1	1	1	1
χ_2	1	-1	1
χ_3	2	0	-1

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- A ”reducible representation” can be written as

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Example

For $\mathbb{Z}_2 \to GL(2, \mathbb{C})$, we would not use $\rho(n) = \begin{bmatrix} 1 & 0 \\ 0 & (-1)^n \end{bmatrix}$

Instead, we use $\rho_1(n) = \begin{bmatrix} 1 \end{bmatrix}$ and $\rho_2(g) = \begin{bmatrix} (\text{-}1)^n \end{bmatrix}$

Irreducible Representations and Conjugacy Classes of S_n

With a lot more work, we can show this theorem

Theorem

There is a bijection between the irreducible representations of the group S_n and the conjugacy classes of S_n.

Remark

There is also a bijection between partitions and conjugacy classes.

Note that two permutations in S_n are conjugate if and only if they have the same cycle structure. From the parts of a partition, we can extract a unique cycle structure. Going the other way, the cycles in a cycle structure can put in order of largest cycle length, thus yielding a partition. So, there is a way of corresponding each irreducible representation of S_n with a partition.

$n! = \sum_{\lambda \vdash n} |\text{SYT of shape } \lambda|^2$

Theorem

The dimension of an irrep equals the number of ways to turn the corresponding partition into a Standard Young Tableaux.

Theorem

The size of a group equals the sum of the squares of the dimensions of its irreps. Informally,

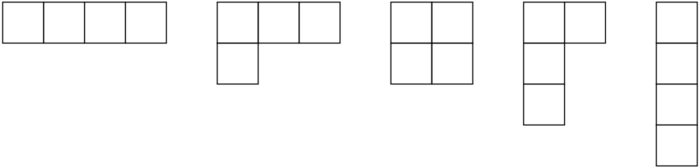
$$|G| = \sum_{\rho} (\dim \rho)^2$$

So, putting this all together, we can get our final theorem:

Theorem

$|S_n| = n! = \sum_{\lambda \vdash n} |SYT \text{ of shape } \lambda|^2$ *where* $\lambda \vdash n$ *means* λ *is a partition of* n.

Example: S_4



1 2 3 4 5

Partition 1: $\begin{array}{|c|c|c|c|}\hline 1&2&3&4\\\hline\end{array}$ Partition 2: $\begin{array}{|c|c|c|}\hline 1&3&4\\\hline 2\\\cline{1-1}\end{array}$ $\begin{array}{|c|c|c|}\hline 1&2&4\\\hline 3\\\cline{1-1}\end{array}$ $\begin{array}{|c|c|c|}\hline 1&2&3\\\hline 4\\\cline{1-1}\end{array}$

Partition 3: $\begin{array}{|c|c|}\hline 1&3\\\hline 2&4\\\hline\end{array}$ $\begin{array}{|c|c|}\hline 1&2\\\hline 3&4\\\hline\end{array}$ Partition 4: $\begin{array}{|c|c|}\hline 1&2\\\hline 3\\\cline{1-1} 4\\\cline{1-1}\end{array}$ $\begin{array}{|c|c|}\hline 1&3\\\hline 2\\\cline{1-1} 4\\\cline{1-1}\end{array}$ $\begin{array}{|c|c|}\hline 1&4\\\hline 2\\\cline{1-1} 3\\\cline{1-1}\end{array}$

Partition 5: $\begin{array}{|c|}\hline 1\\\hline 2\\\hline 3\\\hline 4\\\hline\end{array}$

$$4! = 1^2 + 3^2 + 2^2 + 3^2 + 1^2$$

Acknowledgements

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References

Joe Gallian. *Contemporary Abstract Algebra*, 10th ed. CRC Press, 2020.